# W12/1 Regularity, Krull's Principal Ideal Theorem, DVRs

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**Recall:** For $(R,M)$ local, $M/M^2$ is the cotangent space.

If $R = A(X)_M$ for $X$ a variety, $M = (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n)$ the max. ideal of a point, then $M/M^2 \cong (T_a X)^*$.

If $R$ is noetherian, $\dim_{R/M} M/M^2$ is the minimal number of generators of $M$.

**Thm 7.14** If $R$ is a noetherian ring, $I := (a_1, \dots, a_m) \trianglelefteq R$, and $P$ is a minimal prime over $I$, then $\operatorname{ht} P \leq m$.

Proof later: Krull's Principal Ideal Theorem.

**Cor 7.15:** If $(R,M)$ is a noetherian local ring, then

$$\dim R \leq \dim_{R/M} M/M^2.$$

**Proof:** By L7.13, $M = (a_1, \dots, a_r)$ with $r = \dim_{R/M} M/M^2$.

$\underset{T7.14(1)}{\Longrightarrow} \operatorname{ht} M \leq r \Longrightarrow \dim R \leq r.$ $\square$

**Def:** (1) A local ring $(R,M)$ is regular if $R$ is noetherian and $\underbrace{\dim_{R/M}}_{\text{v.s. dim}} M/M^2 = \underbrace{\dim}_{\text{Krull dim.}} R$.

(2) If $X$ is a variety/a field $K$, $\underline{a} \in X$ is regular if $A(X)_{I(\underline{a})}$ is regular [if $X$ irred, $K = \bar{K}$, then $\underline{a}$ regular $\Leftrightarrow \dim T_{\underline{a}} X = \dim X$]

[$A(X)_{I(\underline{a})}$ is always noetherian]. Otherwise, $\underline{a}$ is singular.

$I(\underline{a})$: max ideal of pt $\underline{a}$

**Exm:** $K$ field, $X \subseteq \mathbb{A}^n$ variety, $I := I(X) = (f_1, \dots, f_r) \trianglelefteq K[x_1, \dots, x_n]$

$$J_{f_1, \dots, f_r} := \left(\partial_{x_j} f_i\right)_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \in K[x_1, \dots, x_n]^{r \times n},$$

Set $A := A(X)$, for $\underline{a} \in X$: $I(\underline{a}) = (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n) \in \operatorname{Max}(A)$.

**Know:** $\underline{a} \in X$ regular $\Leftrightarrow \operatorname{rank} J_{f_1, \dots, f_r}(\underline{a}) = n - \dim(A_{I(\underline{a})})$

**Know:** $\underline{a} \in X$ regular $\Leftrightarrow$ rank $J_{f_1,\dots,f_r}(\underline{a}) = n - \dim(A_{I(\underline{a})})$

Suppose $X$ is irreducible $\Rightarrow$ $A$ domain $\overset{C6.21/P6.23}{\Longrightarrow}$ $\dim(A_{I(\underline{a})}) = \dim A$ $\forall \underline{a} \in X$

Suppose $J_{f_1,\dots,f_r}(\underline{a})$ has full row rank, i.e., $r$,

then:
$$n - r \underset{\substack{\uparrow \\ T7.14 \\ + C6.21/P6.23}}{\leq} \dim A \underset{\substack{\uparrow \\ P6.23}}{=} \dim A_{I(\underline{a})} \underset{\substack{\uparrow \\ \text{Consequence of } T7.14\ (C7.18) \text{ below}}}{\leq} \dim T_{\underline{a}}^{*} X \overset{\substack{\text{Assumption on full rank} \\ \downarrow}}{=} n - r$$

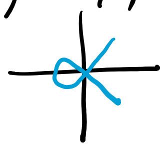
$\Rightarrow$ $\underline{a}$ regular.

If $K = \mathbb{R}$, the implicit function theorem gives, locally around $\underline{a}$, a parametrization of $X$ by a smooth $f: \mathbb{R}^k \supseteq U \to X$.

**Exm:** $\mathbb{A}^2(\mathbb{R})$: $X = V(y - x^2)$ is regular everywhere:
$$J_{y-x^2} = (-2x, \ 1)$$

$X' = V(y^2 - x^2 - x^3)$: $J_{y^2-x^2-x^3} = (-2x - 3x^2, \ 2y)$

vanishes at $(0,0)$, $X'$ is singular there



**Def:** Let $R$ be a ring, $P \in \mathrm{Spec}(R)$, $j: R \to R_P$ localization, $n \geq 0$.
Then $P^{(n)} := j^{-1}(P^n R_P)$ is the $n$-th symbolic power of $P$

**Lemma 7.16** $R$ ring, $P \in \mathrm{Spec}(R)$, $n \geq 0$

(1) $P^{(n)} = \{ a \in R : \exists b \in R \setminus P : ab \in P^n \}$

(2) $P^n \subseteq P^{(n)} \subseteq P$

(3) $P^{(n)}$ is $P$-primary.

**Proof:** (1) $P^{(n)} = \bigcup\limits_{b \in R \setminus P} (P^n : b)$ by P2.1.

(2) $P^n \subseteq j^{-1}(j(P^n)) = P^{(n)}$.

$P^n \subseteq P \Rightarrow P^{(n)} = j^{-1}(j(P^n)) \subseteq j^{-1}(j(P)) \overset{\substack{P \text{ prime} \\ \swarrow}}{=} P$.

(3) $P^n R_P = (P R_P)^n \overset{L5.1}{\Longrightarrow} P^n R_P$ primary $\overset{L5.8}{\Longrightarrow} P^{(n)}$ primary $\square$

**Thm 7.17** (Krull's Principal Ideal Theorem) Let $R$ be a noetherian ring.

(⇐) $I_P$ ($IR_P$) ... $I_P$ primary $\Rightarrow$ $I$ primary $\square$

<u>Thm 7.17</u> (Krull's Principal Ideal Theorem) Let $R$ be a noetherian ring, $a \in R$, $P \in \operatorname{Spec}(R)$ minimal over $(a)$. Then $\operatorname{ht}(P) \leq 1$.

<u>Proof:</u> Suppose $Q' \subseteq Q \subsetneq P$ are prime ideals. <u>Show:</u> $Q' = Q$

Replace $R$ by $R_P / Q'R_P$, then wrt.

$R$ is a local domain w. max. ideal $P$, $Q' = \underline{0}$, $a \in P$, $a \notin Q$

<u>Show:</u> $Q = \underline{0}$.

Consider $Q^{(0)} \supseteq Q^{(1)} \supseteq \ldots$

<u>Claims:</u> (i) $\exists n: \quad Q^{(n)} \subseteq Q^{(n+1)} + (a)$

(ii) $Q^{(n)} = Q^{(n+1)} + PQ^{(n)}$

(i) $R/(a)$ is noeth., $\operatorname{Max}(R/(a)) = \{P/(a)\}$. Since $P/(a)$ is also a min. prime of $R/(a)$, $\operatorname{Spec}(R/(a)) = \{P/(a)\} \Rightarrow \dim R/(a) = 0$

$\xrightarrow{T4.14} R/(a)$ ordinian

$$\Rightarrow Q^{(0)} + (a)/(a) \supseteq Q^{(1)} + (a)/(a) \supseteq \ldots \text{ stabilizes}$$

$$\Rightarrow \exists n: \quad Q^{(n)} + (a)/(a) = Q^{(n+1)} + (a)/(a)$$

$$\Rightarrow Q^{(n)} + (a) = Q^{(n+1)} + (a) \Rightarrow Q^{(n)} \subseteq Q^{(n+1)} + (a).$$

(ii) "$\supseteq$" ✓, "$\subseteq$" Let $b \in Q^{(n)} \xRightarrow{(i)} b = c + ar$, $c \in Q^{(n+1)}$, $r \in R$

$$\left.\begin{array}{l} \Rightarrow ar = b - c \in Q^{(n)} \\ a \notin Q = \sqrt{Q^{(n)}} \end{array}\right\} \xRightarrow{Q^{(n)} \text{ primary}} r \in Q^{(n)}$$

$$\Rightarrow c + ar \in Q^{(n+1)} + PQ^{(n)}.$$

$\square$ (Claims)

$$\xRightarrow{(ii)} Q^{(n)}/Q^{(n+1)} = P \, Q^{(n)}/Q^{(n+1)} \xRightarrow{\text{Nakayama}} Q^{(n)}/Q^{(n+1)} = \underline{0}$$

$$\Rightarrow Q^{(n)} = Q^{(n+1)} \Rightarrow Q^n R_Q = Q^{n+1} R_Q = (QR_Q)(Q^n R_Q)$$

$$\xRightarrow{\text{Nakayama}} Q^n R_Q = \underline{0} \xRightarrow{R \text{ domain}} Q^n = 0 \xrightarrow{R \text{ domain}} Q = 0. \qquad \square$$

$\Longrightarrow Q^n R_Q = \underline{0} \Longrightarrow Q^n = 0 \Rightarrow Q = 0.$ $\square$

**Lemma 7.18** Let $R$ be a noetherian ring, $a \in R$,

$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$ $(n \geq 1)$ a chain in $\mathrm{Spec}(R)$ s.t. $a \in P_n$

$\Rightarrow \exists P_0', \dots, P_{n-1}' \in \mathrm{Spec}(R)$: $P_0' \subsetneq P_1' \subsetneq \dots \subsetneq P_{n-1}' \subsetneq P_n$ and $a \in P_1'$.

**Proof:** Suppose $a \notin P_1$, and let $m$ be maximal s.t. $a \notin P_m$.

Consider $R' := R_{P_{m+1}} / P_{m-1} R_{P_{m+1}}$. Let $P_m' \in \mathrm{Spec}(R)$ s.t. $\overline{P_m'}$ is minimal over $\bar{a} \in R'$.

$\Rightarrow P_{m-1} \subseteq P_m' \subseteq P_{m+1}$,

Since $a \notin P_{m-1}$, $a \in P_m' \rightarrow P_{m-1} \subsetneq P_m'$

$\underline{0} \subsetneq \overline{P_m} \subsetneq \overline{P_{m+1}} \Rightarrow \mathrm{ht}(\overline{P_{m+1}}) \geq 2 \overset{T7.17}{\Rightarrow} \overline{P_{m+1}}$ is not minimal $/ \bar{a}$.

$\Rightarrow P_m' \neq P_{m+1}$.

Now proceed inductively on $m$. $\square$

**Proof of T7.14:** Let $I = (a_1, \dots, a_n) \trianglelefteq R$, $P \in \mathrm{Spec}(R)$ minimal $/ I$.

Show: $\mathrm{ht}\, P \leq n$. Induction on $n$. $n = 0$ ✓

$n-1 \to n$, $n \geq 1$: Let $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_m = P$ in $\mathrm{Spec}(R)$.

$L7.18 \Rightarrow$ wlog $a_1 \in P_1$.

$\Rightarrow P_1/(a_1) \subsetneq P_2/(a_1) \subsetneq \dots \subsetneq P_m/(a_1)$ is a chain in $\mathrm{Spec}(R/(a_1))$,

$P/(a_1)$ is minimal over $I/(a_1) = (\bar{a}_2, \dots, \bar{a}_n)$

$\overset{IH}{\Longrightarrow} \mathrm{ht}\,(P/(a_1)) \leq n-1 \Rightarrow m \leq n \Rightarrow \mathrm{ht}(P) \leq n.$ $\square$

**Cor 7.19** $K$ alg. closed, $X \subseteq \mathbb{A}^n$ a variety with $I(X) = (f_1, \dots, f_r)$

$\Rightarrow \dim X \geq n - r$

**Proof:** $X = X_1 \cup \dots \cup X_t$ with $X_i$ irreducible varieties.

Each $X_i$ corresponds to a minimal prime $P_i \subseteq K[x_1, \dots, x_n]$ over

Each $X_i$ corresponds to a minimal prime $P_i \subseteq K[x_1,\dots,x_n]$ over $I(X) \overset{T7.14}{\Longrightarrow} ht(P_i) \leq r$

$\Longrightarrow \dim X_i = \dim A(X_i) = \dim K[x_1,\dots,x_n]/P_i \overset{P6.23, T6.20}{=} n - ht(P_i) \geq n-r.$ $\square$

# 8. DVRs and Dedekind domains

## 8.1 Discrete Valuations / Let $K$ be a field

**Def:** A discrete valuation is a surjective map $v: K \to \mathbb{Z} \cup \{\infty\}$ s.t.

$\forall a,b \in K$:
$$v(a) = \infty \Leftrightarrow a = 0$$
$$v(ab) = v(a) + v(b) \qquad [\infty + n = \infty = \infty + \infty \quad \forall n \in \mathbb{Z}]$$
$$v(a+b) \geq \min\{v(a), v(b)\}$$

**Exm:** (1) $K = \mathbb{Q}$, $p \in \mathbb{N}$ prime. Each $x \in \mathbb{Q}\setminus\{0\}$ has unique repr.
$$x = p^e \frac{a}{b} \text{ with } e \in \mathbb{Z},\ a,b \in \mathbb{Z}\setminus\{0\},\ p \nmid a,b$$
Then $v_p(x) := e$ $(v_p(0) = \infty)$ is a discrete valuation.

(2) If $K = \mathcal{F}(A)$, $A$ a UFD, each prime element $p$ of $A$ induces a discrete valuation $v_p$ as in (1).

E.g. $\mathbb{Q}(x) \supseteq \mathbb{Q}[x] \ni f$, $v_{x-a}(f)$ = order of vanishing of $f$ at $a$.

**Remark:**
- $v|_{K^\times}: K^\times \to \mathbb{Z}$ is a group epi, in particular $v(1) = 0$, $v(a^{-1}) = -v(a)$ if $a \neq 0$.
- If $a^n = 1$ for some $n \geq 1$, $v(a) = 0$ $[\,0 = v(1) = v(a^n) = n\,v(a) \Rightarrow v(a) = 0]$
- $v(a) = v(-a)$ $[\,v(-a) = \underbrace{v(-1)}_{=0} + v(a) = v(a)]$
- If $v(a) \neq v(b)$, then $v(a+b) = \min\{v(a), v(b)\}$

$[$Wlog. $v(a) < v(b)$: $\underline{v(a+b) \geq v(a)} = v((a+b)-b) \geq \min\{v(a+b), v(b)\}$
$\overset{v(b) > v(a)}{\Longrightarrow} \underline{v(a+b) \leq v(a)}\ ]$

**Def:** Let $v$ be a d.v. on $K$.

- $\mathcal{O}_v := \{a \in K : v(a) \geq 0\}$ is the valuation ring of $v$

**Def:** Let $v$ be a d.v. on $K$.

- $\mathcal{O}_v := \{a \in K : v(a) \geq 0\}$ is the valuation ring of $v$
- $M_v := \{a \in K : v(a) > 0\}$ is the valuation ideal of $v$
- $\mathcal{O}_v / M_v$ is the residue field of $v$
- A uniformizer is an element $\pi \in K$ s.t. $v(\pi) = 1$.

**Exm:** $\mathbb{Z}_{(p)} = \mathcal{O}_{v_p}$, $p\mathbb{Z}_{(p)} = M_{v_p}$ for prime numbers $p$.

**Prop 8.1:** Let $v$ be a d.v. on $K$, $\pi$ a uniformizer.

Then (1) $\mathcal{O}_v$ is a local domain w. max. ideal $M_v$, $\mathcal{O}_v^\times = \{a \in K : v(a) = 0\}$

(2) $M_v = (\pi)$ and the nonzero ideals of $\mathcal{O}_v$ are of the form $M_v^k = (\pi^k)$ for $k \geq 0$.

(3) $\mathcal{O}_v$ is a PID with unique (up to associativity) prime element $\pi$.

In particular: $\mathcal{O}_v = \{\varepsilon \pi^k : k \geq 0, \varepsilon \in \mathcal{O}_v^\times\}$.

**Proof:** (1) Let $a, b \in \mathcal{O}_v \Rightarrow v(ab) = v(a) + v(b) \geq 0$, $v(a+b) \geq \min\{v(a), v(b)\} \geq 0$

$\Rightarrow a+b, ab \in \mathcal{O}_v$. $\quad 0, 1 \in \mathcal{O}_v$ ✓

If $c, d \in M_v$, similarly $c+d \in M_v$. $\quad v(ac) = \underset{\geq 0}{v(a)} + \underset{> 0}{v(c)} > 0 \Rightarrow ac \in M_v$

$\Rightarrow M_v$ is an ideal.

If $a \in \mathcal{O}_v \setminus M_v \Rightarrow v(a^{-1}) = -v(a) = 0$, so $a^{-1} \in \mathcal{O}_v \Rightarrow \mathcal{O}_v \setminus M_v \subseteq \mathcal{O}_v^\times$

$\Rightarrow \mathcal{O}_v$ local w. max. ideal $M_v$.

$0 \neq a \in \mathcal{O}_v^\times \Leftrightarrow a \in \mathcal{O}_v$ and $a^{-1} \in \mathcal{O}_v \Leftrightarrow v(a) = 0$.

(2) Let $0 \neq I \trianglelefteq \mathcal{O}_v$. Let $r := \min\{v(a) : a \in I\} \in \mathbb{N}_0$.

**Claim:** $I = (\pi^r)$.

„$\supseteq$": Let $a_0 \in I$ with $v(a_0) = r$.

$\Rightarrow \pi^r = (a_0^{-1}\pi^r) a_0$, $\quad v(a_0^{-1}\pi^r) = r - v(a_0) = 0 \Rightarrow a_0^{-1}\pi^r \in \mathcal{O}_v$

$\Rightarrow \pi^r \in (a_0) \subseteq I$.

„$\subseteq$" Let $a \in I \Rightarrow v(a) \geq r \Rightarrow v(a\pi^{-r}) = v(a) - r \geq 0$

$\Longrightarrow \Pi^r \in (a_0) = I.$

„$\subseteq$" Let $a \in I \Rightarrow v(a) \geq r \Rightarrow v(a\Pi^{-r}) = v(a) - r \geq 0$

$\Longrightarrow a = \underbrace{(a\Pi^{-r})}_{\in \mathcal{O}_v} \Pi^r \in (\Pi^r)$.

(3) By (2). $\square$